# Advanced Arithmetic from Twelfth-Century Al-Andalus, Surviving Only (and Anonymously) in Latin Translation? A Narrative That Was Never Told.

**Jens Høyrup**

# Abstract

As Ahmed Djebbar has pointed out, eleventh-century and earlier al-Andalus produced a “solid research tradition in arithmetic.” So far, no continuation of this tradition has been known, but analysis of three sections of two Latin works suggest that they borrow material that can hardly come from elsewhere:

1. The *Liber mahameleth*, likely to be a more or less free translation made by Gundisalvi or somebody close to him of an Arabic original presenting “*muʿāmalāt* mathematics vom höheren Standpunkt aus” contains systematic variations, for instance of proportions $\frac{p}{g}$ :: $\frac{P}{G}$ (claimed to deal with prices and goods), where the givens may be sums, differences, products, sums or differences of square roots, etc., solved sometimes by means of algebra, sometimes with appeals to Elements II.5–6, often after reduction by means of proportion techniques.
2. A passage in Chapter 12 of the *Liber abbaci* first presents the simple version of the recreational problem about the “unknown heritage” (likely to be of late Ancient or Byzantine origin): a father leaves to his first son 1 monetary unit and $^1/_n$ of what remains, to the second 2 units and $^1/_n$ of what remains, etc.; in the end, all get the same, and nothing remains. Next it goes on with complicated cases where the arithmetical series is not proportional to 1 – 2 – 3 ..., and the fraction is not an aliquot part. Fibonacci gives an algebraic solution to one variant and also general formulae for all variants – but these do not come from his algebra, and he thus cannot have derived them himself. A complete survey of occurrences once again points to al-Andalus.
3. Chapter 15 Section 1 of Fibonacci’s *Liber abbaci* mainly deals with the ancient theory of means though not telling so. If M is one such mean between A and B, it is shown systematically how each of these three numbers can be found if the other two are given—once more by means of algebra, *Elements* II.5–6, and proportion techniques. The lettering shows that Fibonacci uses an Arabic or Greek source, but no known Arabic or Greek work contains anything similar. However, the structural affinity suggests inspiration from the same environment as produced the *Liber mahameleth*.

So, this seems to be a non-narrative, a story that was not revealed by the participants, and was not discovered by historians so far.

In memory of
*Michel Olsen (1934–2013)*
and
*Gunver Kelstrup (1935–2015)*
dear friends

## A background

The writings of Ibn Rušd, as everybody only slightly familiar with Latin or Hebrew medieval philosophy knows, was a major influence in both. His impact on Arabic thought, on the other hand, was modest—more modest, indeed, than can be explained from his date or from al-Ghazālī's attack on the incoherence of the philosophers (to which he wrote an answer). Similarly, while al-Muʾtaman's eleventh-century *Kitāb al-Istikmāl* still gave rise to further work by Arabic mathematicians [Djebbar 1993: 82 and *passim*], Jābir ibn Aflah's twelfth-century work in astronomy and spherics is much better known from Hebrew and Latin translations than in Arabic [Lorch 1973: 39].

The obvious explanation is that al-Andalus was already cut off from the corresponding courses of Islamic scholarly life. Not all courses, as we know—ibn al-Yāsamīn's work, partly done in al-Andalus and partly in the Maghreb, did survive in Arabic (while leaving no clearly recognizable traces in Latin or in Romance vernaculars); but this work was integrated with an interest that thrived in the vicinity of *madrasah* learning as established in the Maghreb—in the terminology of A. I. Sabra [1987], it represented "naturalized" knowledge. It is therefore tempting to ask whether a fate similar to that of ibn Rušd's philosophy befell some further branches of al-Andalus learning.

One such branch might be theoretical arithmetic. As observed by Ahmed Djebbar in [1993: 86],[1] there was

> in Spain and before the eleventh century, a solid research tradition in

[1] My translation, as all translations in the following with no identified translator.

> arithmetic, whose starting point seems to have been the translation made by Thābit ibn Qurra of Nicomachos's *Introduction to Arithmetic*.

Arabic sources seem to present us with no evidence that this tradition lived on into the twelfth century. However, as it has dawned upon me over the last years, three different Latin and one Tuscan vernacular source which I had worked on without at first seeing any connection between them may be witnesses not only of survival but of impressive expansion. Since not all of what I have written on these occasions can be expressed better (at least not by me, at this moment), what follows will contain a certain amount of auto-plagiarism

## The unknown heritage—the simple version

My first instance is a theoretical elaboration of the solution to a stunning recreational problem—the "unknown heritage"—which I dealt with in [Høyrup 2008].

The standard version of this problem runs as follows: a father leaves to his first son 1 monetary unit and $\frac{1}{n}$ ($n$ usually being 7 or 10) of what remains; to the second he next leaves 2 units and $\frac{1}{n}$ of what remains, etc. In the end all sons get the same amount, and nothing remains. The solution is that there are $n$–1 sons, each of whom receives $n$–1 monetary units. Alternatively the fraction is given first and the arithmetically increasing amount afterwards, in which case $n$–1 sons get $n$ monetary units each.

Our earliest source for both variants of this simple version of the problem is Chapter 12 of Fibonacci's *Liber abbaci* [ed. Boncompagni 1857: 279][2]—"simple" only in comparison with the "sophisticated" version to which we shall return, not in comparison with the other recreational problems dealt with in the same chapter. It is possible to find the only *possible* solution by algebra or by a double false position from the equality of the first two shares, but in order to show that this really *is* a solution one has to perform a stepwise control. It is also possible to find the only

[2] Since all the copious references to the *Liber abbaci* in the following refer to this edition, I shall only indicate them by page number.

possible solution by elementary means from the equality of the last two shares, but this solution suffers from the same defect as the previous one; moreover, it appears to have escaped all medieval and early modern authors presenting the problem, and even all modern historians who have worked on it.[3]

This "simple" problem is found regularly in Italian abbacus books from the early fourteenth century onward, mostly the first variant but also sometimes the second, and the corresponding semi-simple variants where the absolute contributions are not $i$ ($i$ = 1, 2, ...) but $i\varepsilon$ ($i = n, n+1, \ldots$), which corresponds to taking ε and not 1 as the monetary unit and skipping the first $n$–1 heirs. Such semi-simple variants are also presented by Fibonacci. Most of the abbacus authors merely state the solutions, but Jacopo da Firenze offers a full numerical check in his *Tractatus algorismi* from 1307 [ed. trans. Høyrup 2007: 360*f*][4] that the solution is correct (while speaking of orange-picking from a garden instead of inheritance). The *Istratti di ragioni* [ed. Arrighi 1964: 140*f*]—a problem collection from c. 1440 but claiming to go back to Paolo dell'Abbaco (c. 1340) and at least likely to copy material from that period—finds the possible solution by means of a double false position applied to the equality of the first two shares. A number of other abbacus occurrences are listed in [Høyrup 2008: 628–630,

---

3 Since the last son (say, no. $N$) leaves nothing, the remainder $r_N$ of which he takes the fraction $^1/_d$ must be 0 (if not, $(1-{}^1/_d)r_N$ would be left over). But since each visitor picks as many apples as his number before taking $^1/_d$ of the remainder, the $N$th son gets $N$ apples, and so therefore do all the others. But the second-last visitor (no. $N$–1) only picks $N$–1 apples before taking the fraction $^1/_d$ of the remainder $r_{N-1}$. Therefore this fraction must be 1 (he has already picked $N$–1, but should have $N$). Further, he leaves $N$ to the last visitor. In consequence $r_{N-1}$ is $N$+1. $^1/_d$ of this number being 1 ($^{N+1}/_d = 1$), $N$ must be $d$–1.

It is hard to believe that Euler [1774: 488–491] should have overlooked this; but if he has seen it, he does not tell his readers. After all, he is teaching algebra, and may have seen no reason to divulge that algebra is not needed.

4 Since it is immaterial for the present purpose I shall not repeat my arguments that the manuscript Vat. Lat. 4826 most likely is a copy of Jacopo's original treatise, whereas the two other surviving manuscripts descend from a revised version. Who is interested in the question may look at [Høyrup 2007: 6–25].

640*f*] (after 2008 I have noticed quite a few more, none of them offering anything new).

Of particular interest is, on one hand, the appearance of the problem in Maximos Planudes's late thirteenth-century *Calculus according to the Indians, Called the Great* [ed., trans. Allard 1981: 191–194]; and, on the other, its apparent absence from Arabic sources (even though two contain a derived and simplified version).

Planudes presents us with the second occurrence we know about, preceded only by that in the *Liber abbaci*. It follows after the exposition of how to calculate with Hindu-Arabic numerals, coming just before the discussion of the problem to "find a figure equal in perimeter to another figure and a multiple of it in area"—that is, for a given $n$ to find two rectangles[5] ⊏⊐($a$,$b$) and ⊏⊐($c$,$d$) such that $a+b = c+d$, $n{\cdot}ab = cd$ ($a$, $b$, $c$ and $d$ being tacitly assumed to be integers). It serves as illustration of this observation or theorem:[6]

> When a unit is taken away from any square number, the left-over is measured by two numbers multiplied by each other, one smaller than the side of the square by a unit, the other larger than the same side by a unit. As for instance, if from 36 a unit is taken away, 35 is left. This is measured by 5 and 7, since the quintuple of 7 is 35. If again from 35 I take away the part of the larger number, that is the seventh, which is then 5 units, and yet 2 units, the left-over, which is then 28, is measured again by two numbers, one smaller than the said side by two units, the other larger by a unit, since the quadruple of 7 is 28. If again from the 28 I take away 3 units and its seventh, which is then 4, the left-over, which is then 21, is measured by the number which is three units less than the side and by the one which is larger by a unit, since the triple of 7 is 21. And always in this way.

This description does not refer explicitly to pebbles or other counters, but

[5] Χωρίον, here translated "figure," may actually have the more restricted meaning "rectangular area."

The statement of the problem and the first of two solutions are found almost verbatim in the pseudo-Heronic *Geometrica*, Chapter 24 [ed., trans. Heiberg 1912: 414–417].

[6] I try to make a very literal translation from the Greek text.

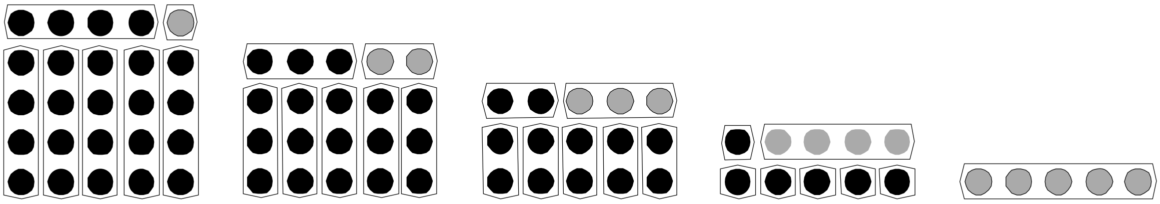

Figure 1

it points rather unambiguously at something like Figure 1 (for simplicity showing a 5×5 square); this is indeed the kind of diagram I made spontaneously when first encountering the problem and the stepwise calculation in Jacopo's *Tractatus*. Even straightforward application of algebraic symbols does not easily show that the procedure goes on in such a way that exactly nothing remains in the end.[7]

The pebble pattern not only allows us to understand the solution, it is also a likely basis for the discovery that the counter-intuitive problem was *possible*. Since pebble arguments were current in Ancient Greek mathematics and in view of the vicinity to material of Ancient or at most of early Byzantine date it is highly plausible that the problem originated in late Greek Antiquity or in a Byzantine context.[8]

The quasi-occurrences in Arabic sources[9] show beyond reasonable doubt that the problem was not transmitted to Fibonacci through Arabic mainstream recreational mathematics. One of them is in ibn al-Yāsamīn's *Talqīḥ al-afkār fī'l ʿamali bi rušūm al-ghubār* ("Fecundation of thoughts through use of *ghubār* numerals"), written in Marrakesh in c. 1190, that is, before the first version of the *Liber abbaci* from 1202. It runs as

7 The proof will have to build on a chain of identities $n{\cdot}(n–p+1) = p+(n+1){\cdot}(n–p)$, $p$ increasing from 1 to $n$.

8 I have observed no traces of the problem or of the underlying theorem in such ancient sources as normally offer veiled references to mathematical practitioners' knowledge or problems (those Platonizing or Pythagoreanizing writers who tried to transform the mathematics they understood into "wisdom," cf. [Høyrup 2001]). It is therefore unlikely that the problem was (widely) known before, say, 200 CE.

9 I thank Mahdi Abdeljaouad for tracing them and supplying me with French translations, which I here translate into English.

follows:

> An inheritance of an unknown amount. A man has died and has left at his death to his six children an unknown amount. He has left to one of the children one dinar and the seventh of what remains, to the second child two dinars and the seventh of what remains, to the third three dinars and the seventh of what remains, to the fourth child 4 dinars and the seventh of what remains, to the fifth child 5 dinars and the seventh of what remains, and to the sixth child what remains. He has required the shares be identical. What is the sum?
>
> The solution is to multiply the number of children by itself, you find 36, it is the unknown sum. This is a rule that recurs in all problems of the same type.

The other comes from the *al-Maʿūna fī ʿilm al-hisāb al-hawāʿī* ("Assistance in the science of mental calculation") written by Ibn al-Hāʾim (1352–1412, Cairo, Mecca and Jerusalem, and familiar with Ibn al-Yāsamīn's work):

> An amount of money has been diminished by one dirham and the seventh [of what remains]; by two dirhams, and then the seventh of what remains; then three dirhams and the seventh of what remains; then four dirhams and the seventh of what remains; then five dirhams and the seventh of what remains. In the end, six remain.
>
> Take the square of the six that remain, it is the amount which was asked for.

Ibn al-Yāsamīn, we see, does not tell the reader that the last share is determined according to the same rule as the preceding ones, and Ibn al-Hāʾim does state that the shares are equal. Both pieces of information are indeed superfluous. The number of shares is given in both versions, and both are "Chinese box problems" that can be solved by backward calculation; none the less, both still use the same rule as Fibonacci's and Planudes' version of the simple problem.

Similar backward calculations could be made for fractions that change and for absolutely defined contributions that are not in arithmetical progression. However, the rule is only valid for a constant fraction $\frac{1}{N+1}$, where $N$ is the given number of shares, and if the absolutely defined contributions are $1+(i–1)$. We are allowed conclude that the Arabic problem descends from the "Christian" problem, and that it is the outcome of an unfelicitous attempt to assimilate it to a more familiar structure.

Mathematicians from the Maghreb or al-Andalus[10] had thus come to know about the problem type already before the *Liber abbaci* was thought of; but their reference to a rule that is adapted to the "Christian" version shows that this latter version with its unknown value of $N$ was not derived from the box-problem versions known in Arabic. Fibonacci and Planudes show us the original, which must none the less antedate ibn al-Yāsamīn and therefore both of them.

## The sophisticated version of the unknown heritage

Fibonacci, however, present us with more, and that is where things become interesting for our topic. A notation will be handy for our further discussion. Division of a number (or an abstract amount of *dragmae*, which in the language inherited from Arabic algebra amounts to the same as Diophantos's *monades*) in such a way that each of the successive parts receives first $\alpha+i\varepsilon$, $i = 0, 1, 2, \ldots$, and afterwards $\phi$ times what remains at hand ($\phi<1$, but not necessarily an aliquot part $\frac{1}{n}$) we shall designate $(\alpha,\varepsilon \mid \phi)$. A division where instead $\phi$ times what is available is taken first, and afterwards an absolutely defined amount $\alpha+i\varepsilon$, we shall designate $(\phi \mid \alpha,\varepsilon)$. In this notation, Fibonacci offers the following problems:

| | | | |
|---|---|---|---|
| $(1,1 \mid \frac{1}{7})$<br>$(\frac{1}{7} \mid 1,1)$<br>$(3,3 \mid \frac{1}{7})$<br>$(\frac{1}{7} \mid 3,3)$ | $(1,1 \mid \frac{2}{11})$<br>$(4,4 \mid \frac{2}{11})$<br>$(\frac{2}{11} \mid 1,1)$<br>$(\frac{2}{11} \mid 4,4)$ | $(2,3 \mid \frac{6}{31})$<br>$(\frac{6}{31} \mid 2,3)$ | $(3,2 \mid \frac{5}{19})$<br>$(\frac{5}{19} \mid 3,2)$ |

The column to the left contains the two variants of the simple version, together with the equally simple variant that the monetary unit is 3 *bizantii* instead of 1. Here, everything is stated in terms of a father distributing

10 Ibn al-Yāsamīn's "all problems of the same type" seems to prove that he was not alone in his area to know about them. Since he had been active in Morocco as well as al-Andalus, this area could be either, or both.

his possessions to his sons.

The remaining columns speak about dividing a number or a number of *dragmae* in the ways indicated. Here, all shares are similarly stated to be equal, with the difference, however, that the last share may be fractional. The problems in the second column are dealt with according to the rule for the simple version, with the unexplained trick that $\frac{2}{11}$ is understood as $\frac{1}{5½}$. Then the number of shares turns out to be 4½, meaning that the last (half-)share is only half of the others.

This trick, however, does not work in the third and the fourth column. For the problem $(2,3\,|\,\frac{6}{31})$ Fibonacci finds the only possible solution by means of *regula recta*, that is, rhetorical first-degree algebra: he calls the number to be divided a *thing*, computes the first two shares and equates them (p. 280). The number to be divided is then found (if expressed in a formula that follows the calculations step by step) to be

$$T = \frac{q^2(\alpha+\varepsilon)-(q-p)q\alpha-(q-p)p\alpha-(\alpha+\varepsilon)pq}{p^2} \; . \tag{1}$$

Fibonacci afterwards makes a step-by-step control calculation, showing that all shares are indeed equal. In the end he claims to extract from the calculation this rul e (*ex hac quidem investigatione talem extraxi regulam*):[11]

$$T = \frac{[(\varepsilon-\alpha)\,q+(q-p)\,\alpha]\cdot(q-p)}{p^2} \; , \tag{2a}$$

$$N = \frac{(\varepsilon-\alpha)\,q+(q-p)\,\alpha}{\varepsilon p} \; , \tag{2b}$$

$$\Delta = \frac{\varepsilon\,(q-p)}{p} \; , \tag{2c}$$

$N$ being the number of shares and $\Delta$ the value of each of them.

This is a case of mild fraud. With techniques at Fibonacci's disposal

[11] Fibonacci expresses himself in terms of the numerical values belonging with his paradigmatic example; but he identifies these so precisely that the translation into symbols in unambiguous.

(1) might be transformed into

$$T = \frac{[q(\alpha+\varepsilon)-(p+q)\alpha]\cdot(q-p)}{p^2} \tag{1*}$$

and possibly into

$$T = \frac{[\varepsilon q-\alpha p]\cdot(q-p)}{p^2} \tag{1**}$$

but never into (2a), however much *we* with our more convenient tools can prove their algebraic identity. We may conclude that Fibonacci took over a rule whose derivation he did not know, and claimed it to be a consequence of his own solution.

This is confirmed by the solution he offers (this time without calculations of his own) for the case $(3,2\,|\,\frac{5}{19})$. Formula (1) would apply smoothly to this case, but formula (2a), which Fibonacci claims to have derived, contains a factor $\varepsilon-\alpha$, which is impossible as long as you have no developed conceptualization of negative numbers; it does not suffice that you know in practice how to make simple additive operations with them, as Fibonacci did. Instead he therefore gives this rule:

$$T = \frac{[(q-p)\alpha-(\alpha-\varepsilon)q]\cdot(q-p)}{p^2}\ , \tag{3}$$

which subtracts $(\alpha-\varepsilon)q$ instead of adding $(\varepsilon-\alpha)q$.

For the case $(\frac{6}{31}\,|\,2,3)$ Fibonacci gives the solution

$$T = \frac{[(\varepsilon-\alpha)\,q+(q-p)\alpha]\cdot q}{p^2} \tag{4}$$

and for $(\frac{5}{19}\,|\,3,2)$

$$T = \frac{[(q-p)\alpha-(\alpha-\varepsilon)q]\cdot q}{p^2}\ , \tag{5}$$

also unconnected to his algebraic calculation.

I know of no surviving text from the fourteenth or the early decades of the fifteenth century where these sophisticated versions of the problem turn up. The first trace after Fibonacci (but a mere trace) is Cardano's treatment of a problem ($\frac{1}{7}$ | 100,100) in the *Practica arithmetice et mensurandi singularis* [1539: fol. FF ii$^{r}$]. Instead of the usual rule, he gives the solution

$$T = \frac{[(q-p)q]\cdot\alpha}{p^2} \tag{6a}$$

which follows from Fibonacci's rules (4) or (5) if we put $\varepsilon = \alpha$ and invert two factors (an inversion which could not be made in the full rules (4) or (5) if $\alpha \neq \varepsilon$). His rule for the number of heirs is

$$N = q-p \ . \tag{6b}$$

Since in (6a) Cardano dutifully finds 1·1 ($p^2$) and divides by it, (6b) must be considered a mistake for

$$N = \frac{q-p}{p} \ . \tag{6b*}$$

Both this mistake and the inversion of factors in (6a) seem to rule out that Cardano built directly on the *Liber abbaci*. In spite of our ignorance it seems likely that there was some kind of circulation of the sophisticated versions.

This is confirmed by Barthélemy de Romans' *Compendy de la praticque des nombres*, a Franco-Provençal treatise from c. 1467,[12] in which these *progressions composées* (as Barthélemy calls them) constitute the high point.[13] Some of the $\phi$-values known from the *Liber abbaci* are repeated by Barthélemy, which has led Spiesser [2003: 156] to find it "very plausible" that Fibonacci was the direct source. Closer analysis shows that the coincidences are very far from being statistically significant [Høyrup 2008:

---

[12] We only possess a revised redaction from 1476 due to Mathieu Préhoude, which has been edited with a modern French translation by Maryvonne Spiesser [2003]. The original treatise was said by Barthélemy [ed. Spiesser 2003: 225] to be written in order to provide clearer understanding to those who had read an earlier work of his,

[13] The pertinent part of the text is found in [Spiesser 2003: 391–423], the translation on pp. 543–579. There is also a substantial commentary (pp. 139–156). My own supplementary discussion is in [Høyrup 2008: 632–638].

635 n. 31]; moreover, when Barthélemy repeats a ϕ-value known from Fibonacci, the appurtenant α and ε are different in 8 out of 9 cases; finally, Barthélemy's global approach as well as his rules are quite different from what we know from Fibonacci, even in that part of the text where Barthélemy seems to borrow from precursors (namely [Spiesser 2003: 391–402]).

If Fibonacci's own text shows him to have borrowed (which he does not deny in general, even though here he pretends to have extracted the rules from his own calculation), and if both Cardano's hear-say knowledge and Barthélemy's real acquaintance with the sophisticated version of our problem do not depend on the *Liber abbaci*—then where does it come from?

Fibonacci himself tells (p. 1) that he had studied abbacus matters, first (when a boy) in Bejaïa, and afterwards in "Egypt, Syria, Greece [i.e., Byzantium], Sicily and Provence." Above I argued that the simple version is likely to have originated in Byzantium or in later Greek Antiquity. There is also evidence that Fibonacci himself learned about that version in Byzantium. His heritage problems speak about *bizantii*, and every time this coin is mentioned together with a place in the *Liber abbaci* (six times in total), this place is Constantinople; the correlation holds the other way too, Byzantium problems never speak about other coin. Either Fibonacci's choice of *bizantii* is simply due to what was in the problem he borrowed, or he wanted to intimate that this was known to him as a Byzantine problem.

However, neither Planudes nor any other Byzantine source knows the sophisticated version, and Fibonacci has no *bizantii* (nor any coin except the almost-abstract *dragma*) in the three corresponding columns. Ibn al-Yāsamīn's downgraded version of the simple problem shows on the other hand that this latter version cannot have been part of the general tradition of Arabic recreational mathematics.

That leaves us with only two possible locations for the creation of the sophisticated version: Provence, or the Iberian world, to which Provence had connections. In [2008: 632] I explicitly denied the presence of any version of the problem in the Castilian *Libro de arismética que es dicho alguarismo* [ed. Caunedo del Potro & Córdoba de la Llave 2000: 133–213] from 1393, which in several respects shows more traces of the heritage from al-Andalus than do Catalonian and Provençal writings. For that reason I tended to point to Provence, although with strong doubts because of the lack of any evidence that twelfth-century Provence should have been the home of mathematicians with the required level of competence. As it turns out, I was mistaken—both $(1,1\,|\,\frac{1}{10})$ and $(1,1\,|\,\frac{1}{11})$ are found on p. 169. On the other hand, no instance of any version is found in known Provençal writings antedating the mid–fifteenth-century *Traicté de la praticque d'algorisme* (which is somehow connected to Barthélemy's *Compendy* and may even be the earlier treatise which he tells to elucidate, cf. note 12[14]). All in all an Iberian invention therefore now seems much more plausible than a Provençal origin.

How could the invention have been made? Were the required tools at hand for those who know to wield them?

Once it is known that a solution can be found, an algebraic solution like the one produced by Fibonacci is of course possible. However, it can never provide the idea that the problems should have a solution. It is no doubt possible to guess, on the basis of the simple version, that other arithmetical series and other fractions will work more or less in the same way, and then try. But a direct proof can also by made by means of a tool that Fibonacci uses elsewhere and must have learned in some place: the line diagram.

We may look at Figure 2, supposing successive shares to be found by taking first absolutely defined amounts $a_n$, and then a fixed fraction of what remains. $AB$ represents $S_\mathrm{n}$, that is, the amount that is at disposition when the $n$-th share is to be taken, $n$ being arbitrary (but possible). This share is $AD$, consisting of $AC = a_n$ and $CD = \phi CB$. The following share is $DF$, consisting of $DE = a_{n+1}$ and $EF = \phi EB$. Since $AD = DF = \Delta$, $CB = CD+DB$, and $EB = EF+FB$, we find that

$$a_{n+1}-a_n = \phi(CB-EB) = \phi(CD-EF)+\phi(DB-FB) = \phi(a_{n+1}-a_n)+\phi\Delta\ ,$$

whence

[14] I thank Stéphane Lamassé for giving me acccess to the transcription of the *Traicté* in his unpublished dissertaion.

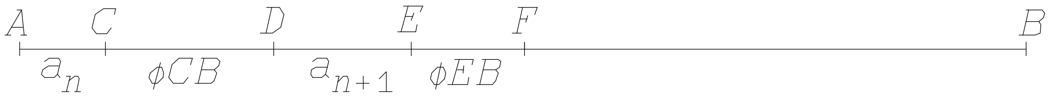


**Figure 2**

$$(1-\phi)\cdot(a_{n+1}-a_n) = \phi\Delta$$

and further (in order to avoid a formal algebraic division) the proportion

$$\Delta : (a_{n+1}-a_n) :: (1-\phi) : \phi\ .$$

Euclid's *Data*, prop. 2 [trans. Taisbak 2003: 254] states that "If a given magnitude [here $\Delta$] have a given ratio [here $(1-\phi):\phi$] to some other magnitude [here $a_{n+1}-a_n$], the other is also given in magnitude" (down-to-earth application of the rule of three is also possible). Therefore $a_{n+1}-a_n$ is constant, and the successive $a_i$ constitute an arithmetical progression; if they do, on the other hand, all shares will be equal (all steps are linear and therefore reversible).

There can be no doubt that this argument *could* have been made in eleventh-century al-Andalus—and why not then in the twelfth century too, before the final catastrophe?

## *Liber mahameleth*

We shall return to the *Liber abbaci*, but first we shall look at the *Liber mahameleth*, an anonymous Latin work that was discovered by Jacques Sesiano in 1974 and first described by him in [1988]; in recent years, two critical editions have appeared, prepared respectively by Anne-Marie Vlasschaert [2010] and Sesiano [2014]. Both editors agree that the title refers to Arabic *al-muʿāmalāt*, which seems indeed beyond dispute. Sesiano [2014: xviii*f*] proposes the compilation to have been made by John of Seville, primarily because a fifteenth-century abbacus book refers to the author as *ispano*, "Spanish," which Sesiano takes as a mistake for *hispalensis*, "Sevillian," but also because the *Liber mahameleth* shares a number of passages with John's *Liber algorismi*. He further points to evidence that the work must have been written in an Arabic environment, that is, before John supposedly went to Toledo. Sesiano himself raises two objections.

Firstly, the Latin of the *Liber mahameleth* is much more polished than in other writings ascribed to John; secondly, the work seems never to have received the final touch, which agrees badly with a work produced by somebody who still had many years of activity waiting for him.

On this point, Charles Burnett's work [2002] on the identity of John becomes relevant. As it turns out, a group of translations from the Arabic, mostly (perhaps all) from the 1130s and 1140s, are attributed to a Johannes Hispalensis; another group is attributed to one Johannes Hispanus / Hispanis, John of Spain, who was connected to Dominicus Gundisalvi (thus plausibly working in and in any case linked to Toledo), active around the 1150s, and who may but need not be identical with John of Seville. This is the John who was responsible for the *Liber algorismi*. Burnett points out that the passages which the *Liber mahameleth* shares with the *Liber algorismi* are also shared with Gundisalvi's *De divisione philosophiae*, to which it is indeed even closer (Burnett observes further echoes in *De scientiis*, Gundisalvi's translation of al-Fārābī's *On the Classification of the Sciences*). Burnett concludes (p. 70) that the "impression that one gets, therefore, is that the *Liber mahameleth* is written either by Gundisalvi or by a close associate of his"—and in the latter case Burnett opts for John (of Spain, possibly but not necessarily the same as John of Seville).

Vlasschaert [2010: 30] tends to see Gundisalvi as the compiler, though with some reservation. To this an observation can be added which seems to rule out John. The *Liber mahameleth* makes consistently use of the algebraic terms *census* and *res*, which were to become the standard terminology of Latin algebra: (since the systematic introduction to algebra which the *Liber mahameleth* refers to regularly has been lost, there is no occasion to introduce the *radix* as root of the *census*). The second part of the *Liber algorismi*, on the other hand, contains a small "excerpt of the book called *gebla mucabala* [ed. Burnett, Zhao & Lampe 2007: 163–165], and here *res*, not *census* is used as translation of *māl*. Given the otherwise close relations between the *Liber mahameleth* and the *Liber de algorismi*, this seems to exclude common authorship. More directly it excludes Sesiano's hypothesis that the *Liber mahameleth* was written by John while he was still in Muslim area; another argument against this hypothesis is that the *Liber de algorismi* makes use of other translations circulating in the Toledan area [Burnett 2002: 84].

However, Sesiano's evidence that the *Liber mahameleth* (or at least the bulk) was written in Muslim area remains strong, while the mathematics of the book clearly betrays the hand of a practising competent mathematician—as we shall see, at least as familiar with proportion theory as with *al-jabr*, and therefore belonging to the class of mathematicians who integrated mathematics and astronomy.[15] Neither Gundisalvi nor his translator friends belonged to that category—their competence in the field, when not mainly metamathematical or philosophical, was rather elementary. Moreover, Gundisalvi's *De divisione philosophiae* [ed. Baur 1903: 93] speaks of a "book which in Arabic is called *Mahamalech*,"[16] and lists its contents in almost the same words as those in which the *Liber mahameleth* [ed. Vlasschaert 2010: 7] presents its contents. In summary it is therefore almost certain that the *Liber mahameleth* was compiled in Muslim area by an astronomer-mathematician—and almost as certain that the main stem was no only compiled but authored. Gundisalvi may still have made adjustments—as can be seen if one compares his free translation of al-Fārābī's *On the Classification of the Sciences*, characterized by deletions as well as additions, with the Arabic text and with the translation prepared by Gerard of Cremona (all in [Gonzales Palencia 1953]).[17]

---

15 We may speak of these as "astronomer-mathematicians"—to be distinguished from the class of "jurisprudent-mathematicians", those who wrote about the kind of mathematics that might be needed by legal scholars—examples of the latter category are ibn Thabāt [ed. Rebstock 1993] and ʿAlī ibn al-Ḫiḍr [ed. Rebstock 2001].

16 All manuscripts used by Ludwig Baur go back to a single "probably already secondary and error-ridden" archetype [Baur 1903: 154]. It is quite possible that the original had *Liber mahamaleth*, and that a copyist read a *t* as *c*. Conveniently, however, the spelling with *c* allows us to distinguish this Arabic treatise from the Latin compilation.

17 We may thus say that the ascription of the *Liber mahameleth* to John or to

Of particular interest is the way commercial problems (those belonging with genuine *muʿāmalāt*-mathematics) serve as pretext for advanced experimentation. We may look at those derived from problems of buying and selling [ed. Vlasschaert 2010: 193–211]—those whose properly practical version would be solved by the rule of three or one of its cognates (those where division precedes multiplication).[18] I omit the detailed calculations for simple variants where, for instance, addition is replaced by subtraction and the operations used for the solution changed correspondingly (for example, from transformation *conjunctim* to *disjunctim*[19]) but no more. Such variation is made systematically throughout the *Liber mahameleth*.

If $p$ and $P$ stand for prices and $q$ and $Q$ for the appurtenant quantities, we have $\frac{q}{p}::\frac{Q}{P}$ (this is meant as a proportion, not an equation involving two fractions[20]). The start of each problem is indicated by $\text{page}_{\text{line}}$.

---

Gundisalvi, though made by colleagues who are highly respectful of Arabic science, represents a narrative which tacitly presupposes that only Christians (perhaps Christians living in Muslim area) could get the idea to submit practical knowledge to the inquisitive eye of theory. For a strongly contrasting opinion, cf. [Høyrup 1987].

18 A more complete presentation of this problem group is forthcoming in the proceedings of the "11ième Colloque Maghrébin sur l'histoire des mathématiques arabes, École normale Supérieure, Kouba – Alger, 26, 27, 28 octobre 2013." Since at that occasion I used Vlasschaert's edition, and since I also reviewed her volume [Høyrup 2015] and therefore have a heavily annotated copy ready for use, my page references will be to her edition.

19 For ease of reference, also in what follows, I list the full set of operations on a proportion $\frac{a}{b} :: \frac{c}{d}$ as given in Campanus' version of the *Elements* [ed. Busard 2005: 171*f*]:

| | | | |
|---|---|---|---|
| *e contrario:* | $\frac{b}{a} :: \frac{d}{c}$ | *conversa:* | $\frac{a}{a+b} :: \frac{c}{c+d}$ |
| *permutata:* | $\frac{a}{c} :: \frac{b}{d}$ | *eversa:* | $\frac{a}{a-b} :: \frac{c}{c-d}$ |
| *conjuncta:* | $\frac{a+b}{b} :: \frac{c+d}{d}$ | *aequa:* | $\frac{a}{b} :: \frac{a+c}{b+d}$ |
| *disjuncta:* | $\frac{a-b}{b} :: \frac{c-d}{d}$ | | |

20 Evidently, this and similar expressions are not proportions in the classical sense (quantities and prices have different dimensions); still, the text not only handles them as if they were but also states explicitly in the beginning of the chapter on buying and selling (p. 186) that the *proportio* of the first quantity to its price is as

$193_7$ $\frac{3}{13}::\frac{Q}{P}$, $Q+P = 60$. This is solved by means of proportion theory, namely via transformation into $\frac{3}{3+13}::\frac{Q}{Q+P}$ and subsequent use of the rule of three.

$193_{32}$ $\frac{3}{13}::\frac{Q}{P}$, $P–Q = 60$. Similarly.

$194_{13}$ $\frac{3}{8}::\frac{Q}{P}$, $Q{\cdot}P = 216$. Fractions (or the rules that $\frac{P}{Q} :: \frac{PQ}{QQ} :: \frac{PP}{PQ}$) are not mentioned, but the solution that is offered builds on awareness that

$$(3{\cdot}216)\div 8 = \tfrac{3}{8}{\cdot}216 = \tfrac{Q}{P}{\cdot}(Q{\cdot}P) = Q^2$$

and

$$(8{\cdot}216)\div 3 = \tfrac{8}{3}{\cdot}216 = \tfrac{P}{Q}{\cdot}(Q{\cdot}P) = P^2$$

$194_{27}$ $\frac{4}{9}::\frac{Q}{P}$, $\sqrt{Q}+\sqrt{P} = 7\frac{1}{2}$. Uses but does not make explicit that $\frac{\sqrt{4}}{\sqrt{9}}::\frac{\sqrt{Q}}{\sqrt{P}}$, which is no standard theorem from the theory of proportions[21] but follows easily from an arithmetical understanding. From here as at $193_7$.

Alternatively,

$$\sqrt{\tfrac{4}{9}}+1 = \frac{\sqrt{Q}}{\sqrt{P}}+1 = \frac{\sqrt{Q}+\sqrt{P}}{\sqrt{P}} = \frac{7\frac{1}{2}}{\sqrt{P}} ,$$

which also presupposes an underlying arithmetical conceptualization.

Yet another alternative makes the claim that

$$\left(\sqrt{\frac{(\sqrt{P}+\sqrt{Q})^2}{(P-Q)\,/\,Q}+\left(\frac{\sqrt{P}+\sqrt{Q}}{(P-Q)\,/\,Q}\right)^2} - \frac{\sqrt{P}+\sqrt{Q}}{(P-Q)\,/\,Q}\right)^2 = Q ,$$

– which is true but not easy to see or even verify, in particular not if not expressed in modern symbolism. The text does not explain.

$195_{17}$ $\frac{4}{9}::\frac{Q}{P}$, $\sqrt{P}–\sqrt{Q} = 1\frac{1}{2}$. All three analogous procedures are shown.

$196_1$ $\frac{4}{9}::\frac{Q}{P}$, $\sqrt{Q}{\cdot}\sqrt{P} = 24$. It is tacitly presupposed once again that $\frac{\sqrt{4}}{\sqrt{9}}::\frac{\sqrt{Q}}{\sqrt{P}}$. The problem is therefore analogous to the one at $194_{13}$. However, the first solution that is offered is

$$\frac{\sqrt{P}{\cdot}\sqrt{Q}}{\sqrt{4}{\cdot}\sqrt{9}}{\cdot}4 = Q , \quad \frac{\sqrt{P}{\cdot}\sqrt{Q}}{\sqrt{4}{\cdot}\sqrt{9}}{\cdot}9 = P ,$$

which suggests awareness that the initial proportion means that $Q = 4s$, $P = 9s$ with some shared factor $s$. In the problem at $201_{10}$ an explicit geometric argument for an analogous insight is given.

---

that of the second quantity to its price. We must think of proportions *between the measuring numbers in the given units*—then the dimension problem is eliminated.

[21] The reverse, when formulated as dealing with the composition of ratios, certainly is.

Alternatively, a procedure related to that at $194_{13}$ is suggested.

Finally, it is proposed to multiply 24 by itself, which yields *PQ*. Then (as explained) the problem is *strictly* analogous to that at $194_{13}$.

A chapter follows “about the same, with [algebraic] things.” When *res* and *census* appear in the text I shall render them by $r$ and $C$, respectively.

$196_{14}$ $\frac{3}{10+r} :: \frac{1}{r}$.[22] This is transformed into $\frac{3}{10+r} :: \frac{3}{3r}$, whence $3r = 10+r$, which is solved in the usual *al-jabr* way. Alternatively, the proportion is transformed into $\frac{3-1}{(10+r)-r} :: \frac{1}{r}$, that is, $\frac{2}{10} :: \frac{1}{r}$, whence $\frac{1}{5} :: \frac{1}{r}$, etc. As we see, cross-multiplication is not used to establish the equation; instead the antecedents are made equal, whence the consequents also become equal. This preference is general.

$196_{26}$ $\frac{4}{20+2r} :: \frac{1½}{2r+3}$. Through multiplication of the right-hand terms by $4 \div 1\frac{1}{2} = 2\frac{1}{3}$, this is transformed into $\frac{4}{20+2r} :: \frac{4}{5\frac{1}{3}r+8}$, whence $5\frac{1}{3}r+3 = 20+r$, etc.

Alternatively: $\frac{4}{20+2r} :: \frac{1½}{2r+3} :: \frac{2½}{17}$. But $1\frac{1}{2} \div 2\frac{1}{2} = \frac{3}{5}$, whence $\frac{1½}{2r+3} :: \frac{1\frac{1}{2}}{\frac{3}{5} \cdot 17}$, etc. It is pointed out that this ruse only works because we have the same multiple of $r$ left and right.

$197_{15}$ $\frac{8}{20+r} :: \frac{2}{r-1}$. Transforming we find $\frac{2}{r-1} :: \frac{6}{21} :: \frac{2}{7}$, whence $r-1 = 7$, etc.

Alternatively, since $8 \div 2 = 4$, $\frac{8}{20+r} :: \frac{8}{4r-4}$, etc.

$197_{33}$ $\frac{6}{10+r} :: \frac{2}{r}$. Transformed into $\frac{2}{r} :: \frac{4}{10} :: \frac{2}{5}$, whence $r = 5$.

$198_{4}$ $\frac{6}{10+r} :: \frac{2}{r+1}$. By transformation $\frac{2}{r+1} :: \frac{4}{9}$, etc.

$198_{14}$ $\frac{3}{20+r} :: \frac{\frac{q}{2}}{\frac{2}{3}r-2}$, which is transformed into $\frac{3}{20+r} :: \frac{3}{4r-12}$, etc.

$198_{24}$ $\frac{6}{10-r} :: \frac{2\frac{2}{3}}{r}$. By transformation $\frac{2}{r} :: \frac{8}{10}$, whence $\frac{8}{4r} :: \frac{8}{10}$, etc.

An alternative that does not depend on the presence of precisely *one* thing left and right transforms the proportion into $\frac{6}{10-r} :: \frac{6}{3r}$, etc.

$198_{34}$ $\frac{4}{8-r} :: \frac{2}{r+1}$. First solved via transformation into $\frac{6}{9} :: \frac{2}{r+1}$, which should give $\frac{2}{3} :: \frac{2}{r+1}$ but by error becomes $\frac{2}{3} :: \frac{2}{r}$, whence $r = 3$. Then, as in the previous example, by the more generally valid alternative, which gives the correct result $r = 2$. The discrepancy is not

[22] “Three measures are given for 10 coins and a thing, but this thing is the price of one measure.” Similarly in the following questions.

discussed and thus probably not noticed. Could this be evidence that two different hands are involved?

$199_{11}$ $\frac{4}{20-2r} :: \frac{1½}{2r-3}$. Solved by the "general" method of the previous two examples. In the end it is pointed out that this can only be understood if one has studied algebra or Euclid's book, "which however have been sufficiently explained."

In "another chapter about an unknown in buying and selling" then follows:

$199_{27}$ An unknown number of measures is sold for 93, and addition of this number to the price of one measure gives 34—in our symbols (since no *res* occurs): $x+\frac{93}{x} = 34$. At first the solution is given as $\frac{34}{2} \pm \sqrt{(\frac{34}{2})^2-93}$ , the sign depending on whether the number of measures exceeds or falls short of the price of one measure. Next a geometric argument based on the principles of *Elements* II.5 is given. Euclid is not mentioned, however, which the compiler-author is elsewhere fond of doing; since the argument uses a subdivided line only, the direct inspiration might be Abū Kāmil's similar proof for the fifth *al-jabr* case (possession plus number equals things) [ed. Rashed 2013: 260–263]—elsewhere it is clear that the author knew Abū Kāmil well.

$200_{27}$ The first of the two corresponding subtractive variants, namely the one in which the number of measures subtracted from the price of one of them gives 28. First a numerical prescription is given, next a line-based geometric proof. If instead (the second subtractive variant) subtraction of the number of measures from the price of one of them gives 28, one should proceed correspondingly.

$201_{10}$ $\frac{q}{p} :: \frac{Q}{P}$, $pq = 6$, $PQ = 24$, $(p+q)+(P+Q) = 15$. Once again the argument appears to go via the factor of proportionality $s$, $sp = P$, $sq = Q$—as this time a geometric argument confirms. At first $s$ is found as $\sqrt{\frac{PQ}{pq}} = 2$. Therefore $p+q = \frac{1}{1+2} \cdot (p+q+P+Q) = \frac{1}{1+2} \cdot 15 = 5$. Since we already know $pq$, we can proceed according to the fifth case of *al-jabr* or *Elements* II.5, none of which are mentioned; the double solution is, however.

$202_{34}$ $\frac{q}{p} :: \frac{Q}{P}$, $pq = 10$, $PQ = 30$, $(p+q)+(P+Q) = 20$. This seemingly innocuous variation of the preceding question leads to an irrational value $s = \sqrt{3}$, and therefore to complications and a cross-reference to the chapter about roots (where indeed the necessary explanations are found). In the end, this leads to a discussion in terms of the classification of *Elements* X (not mentioned here, which suggests that these classes are supposed to be familiar—elsewhere the book *is* mentioned).

$204_{24}$ $\frac{q}{p}::\frac{Q}{P}$, $pq = 6$, $PQ = 24$, $(P+Q)–(p+q) = 5$. The first part of this subtractive variant of the problem at $201_{10}$ is a prescription analogous to the one for the additive variants; for the second part, a mere cross-reference is given.

$204_{35}$ $\frac{q}{p}::\frac{Q}{P}$, $pq = 6$, $PQ = 24$, $(p+q)\cdot(P+Q) = 10$. Without being identified, the proportionality factor $s$ is found as $\sqrt{\frac{PQ}{pq}}$ ; next (since $P+Q = s(p+q)$, which also is not explained) $p+q$ is found as $\sqrt{\frac{(p+q)\cdot(P+Q)}{s}}$ . For the rest, a cross-reference is given. For the first step, however, a geometric demonstration is supplied in the end.

$205_{21}$ $\frac{q}{p}::\frac{Q}{P}$, $pq = 20$, $PQ = 10$, $(p+q)\cdot(P+Q) = \sqrt{5760}$. This is explained to follow the previous question, but evidently gives rise to complicated manipulations of roots, for which reason both ways to solve the problem are discussed in detail.

$207_{2}$ $\sqrt{p} = 3q$, $p–q = 34$ (the identification of the two numbers as price and appurtenant quantity is irrelevant). The solution follows from a quadratic completion ($\sqrt{p} = t$, $q = \frac{1}{3}t$):

$$t^2–\frac{1}{3}t = 34$$

$$t^2–2\cdot\frac{1}{6}t+(\frac{1}{6})^2 = 34\frac{1}{36}$$

$$t–\frac{1}{6} = 5\frac{5}{6}$$

$$t = 6$$

At first a purely numerical prescription is given, afterwards follows a geometric, line-based proof—that is, the author avoids the automatic procedures of *al-jabr*, apparently preferring what is based on more respectable principles.

$207_{24}$ $\sqrt{p} = 2q$, $p+q = 18$. This additive counterpart of the preceding problem is solved analogously

$208_9$ $\frac{6}{4+r} :: \frac{2}{3\sqrt{(4+r)}}$, which is transformed into $\frac{6}{4+r} :: \frac{6}{9\sqrt{(4+r)}}$. The resulting equation $(4+r) = 9\sqrt{4+r}$ is not stated explicitly, but the numerical prescription corresponds to its transformation into $\sqrt{4+r} = 9$ and further into $4+r = 81$.

$208_{15}$ $\frac{6}{4-r} :: \frac{2}{3\sqrt{(4-r)}}$. Solved correspondingly.

$208_{21}$ $\frac{3}{x+y} :: \frac{1}{y+\frac{1}{9}y}$, $xy = 21$ ($x$ and $y$ stand for the "two different things" of the text). The prescription corresponds to the transformation of the proportion into $\frac{3}{x+y} :: \frac{3}{3y+\frac{1}{3}y}$, whence $x+y = 3\frac{1}{3}y$, $x = 2\frac{1}{3}y$, $2\frac{1}{3}y^2 = 21$, $y^2 = 9$, and finally $y = 3$, $x = 7$. After the prescription comes a line-based argument corresponding to these symbolic equations.

Alternatively, the problem can be solved "according to algebra." Here, the *thing* ($r$) takes the place of $y$, while the *dragma* ($d$) takes that of $x$. This time, the equation comes from a different but similar transformation of the proportion, namely into $\frac{1}{\frac{1}{3}d+\frac{1}{3}r} :: \frac{1}{r+\frac{1}{9}r}$. From here follows the equation $r+\frac{1}{9}r = \frac{1}{3}d+\frac{1}{3}$r, and therefore $d = 2\frac{1}{3}r$. Inserting this in $rd = 21$ we get $2\frac{1}{3}C = 21$, $C = 9$, $r = 3$.

$209_{30}$ $\frac{5}{x+y} :: \frac{1}{\frac{1}{3}x+2}$, $xy = 144$. Both methods of the previous problem are applied, now leading to mixed second-degree problems; the line-argument goes through the complete calculation, whereas the algebraic solution satisfies itself with the first step and the explanations that "the rest is done as we have taught in the algebra."

The text goes on with other classical *muʿāmalāt* topics (profit and interest, partnership, etc.). Also here we first find the proper commercial problems, and afterwards systematic explorations of the sophisticated problems to which the topics can give rise. Obviously, at least this part of the treatise (including its the algebra chapter, now lost, to which it refers, and the discussion of roots) is one piece, not put together from disparate books; and obviously, all of it is far beyond the mathematical level of Gundisalvi and his circle, including the John who compiled the *Liber de algorismi*. Nor is it in proper *al-jabr* style, with its ample recourse to proportion techniques and *Elements* II—we must rather suppose the author to be an astronomer-mathematician presenting *muʿāmalāt*-mathematics *von* *höheren Standpunkt aus.*

## Back to Fibonacci

We shall now return to the *Liber abbaci*, but to Chapter 15. This chapter falls into three sections, all relevant to our purpose, but most relevant Section 1.[23]

Chapter 15 as a whole is introduced as treating of "geometrical rules, and questions of algebra and almuchabala," while Section 1 is said to deal with "proportions of three and four quantities, to which many questions pertaining to geometry are reduced" (p. 387).[24] Actually, the succeeding text speaks consistently of *numbers*, not quantities; moreover, the results are not always used in the ensuing "geometry" section when they would serve. Fibonacci's initial announcement is somewhat out of keeping with what follows.

When three numbers are involved, Fibonacci refers to them as minor / middle / major; as first / middle / major; or as first / second / third. If four, as first / second / third / fourth. For convenience, I shall use $P/Q/R$ respectively $P/Q/R/S$. Mostly, the problems are accompanied by letter-carrying lines drawn in the margin. First come proportions involving three number, afterwards a few questions involving four numbers. Using conjunction, disjunction, permutation etc., Fibonacci transforms the given proportion in such a way that the numbers can be found from the product rules by means of addition or subtraction or, more often, *Elements* II.5–6.

---

[23] I first worked on this in the context of a general investigation of Fibonacci's references to "proportions" [Høyrup 2011].

[24] In general, Fibonacci speaks of a "proportion" (*proportio*) both where we would do so (and where many other Medieval authors would write *proportionalitas*) and where we would see a "ratio." I shall not try to impose modern distinctions on Fibonacci; even modern usage is indeed inconsistent, failing for instance to distinguish ratios (relations between integers) from fractions (rational numbers).

In symbolic language, I shall write the ratio between $a$ and $b$ as $\frac{a}{b}$, corresponding to the way I write proportions, and the corresponding fraction as ${}^{a}\!/_{b}$.

Fibonacci never refers to Euclid here, as is his habit elsewhere,[25] but only uses line diagrams. Since the omission is systematic, we may already at this point be confident that his use is indirect and the material thus borrowed.

The section can be divided into 50 propositions, most of which are problems[26]—in overview (page numbers are given in superscript; Fibonacci's headings are indicated as "---*heading*---"; divisions "------" correspond to simple paragraph divisions in the edition):

> #1–3 deal with three numbers in continued proportion, $P:Q:R$ (that is, $\frac{P}{Q} :: \frac{Q}{R}$) of which one and the sum of the other two are given. The naming of segments presupposes the Latin alphabetic order *a*, *b*, *c*, ….

---*Incipit pars prima*---

#1[(387)] $P+Q = 10$, $R = 9$. *Conjunctim* $\frac{P+Q}{Q} :: \frac{Q+R}{R}$, whence *Elements* II.6 can be applied to $Q\cdot(Q+9) = 90$.

#2 $P = 4$, $Q+R = 15$. Analogous.

#3 $Q = 6$, $P+R = 13$. The product rule gives $P\cdot R = 36$, which is transformed so as to permit use of *Elements* II.6 (direct use of II.5 would seem obvious).

------

> #4–38 still treat of three numbers, but now differences between the numbers are among the given magnitudes. The alphabetic order changes to *a*, *b*, *g*, *d*, …, pointing to use of a Greek or an Arabic source. However, in #4–5, still dealing with a continued proportion, *c* is made use of in the calculations:

#4[(388)] $P:Q:R$, $Q–P = 2$, $R = 9$. *Disjunctim* $\frac{R}{Q} :: \frac{R–Q}{Q–P}$. Solved by means of *Elements* II.5.

#5 $P:Q:R$, $R–P = 5$, $Q = 6$. The product rule gives $P\cdot R = 36$, which allows use of *Elements* II.6.5.

#6 An aside which explains that $\frac{a}{b} :: \frac{c}{d}$ entails that the squares of the numbers are also in proportion—a proportion which can then be

[25] See [Folkerts 2006, article IX].

[26] The numbering is mine. Fibonacci has none.

transformed *conjunctim*, *e converso* etc. Further, that the same holds for the cubes. This is no consequence of what precedes nor a preparation for what follows immediately (neither a corollary nor a lemma); and when it is eventually used in #50 there is no backward reference. We may think of note 20, above.

---*Modus alius proportionis inter tres numeros*---

#7(389) $\frac{R-Q}{Q-P} :: \frac{R}{P}$, $Q$ unknown. $R-P$ thus has to be split into two parts having the ratio $R{:}P$; this is solved as a partnership problem (the link is not made explicit).

#8 Same proportion, $R$ unknown. *Permutatim* $\frac{R}{R-Q} :: \frac{P}{Q-P}$, a first-degree problem.

#9 Same proportion, $P$ unknown, solved similarly.

---*Modus alius proportionis inter tres numeros*---

#10 $\frac{Q-P}{R-Q} :: \frac{R}{P}$, $Q$ unknown. *Conjunctim* $\frac{(Q-P)+(R-Q)}{R-Q} :: \frac{R+P}{P}$, a first-degree problem.

#11(390) Same proportion, $R$ unknown. Product rule, and *Elements* II.6.

#12 Same proportion, $P$ unknown. Analogously.

---*Modus alius proportionis in tribus numeris*---

#13 $\frac{R}{P} :: \frac{(R-Q)+(Q-P)}{R-Q}$, $Q$ unknown. Since $(R-Q)+(Q-P) = R-P$, this is as simple first-degree problem.

#14 Same proportion, $R$ unknown. $\frac{R-P}{P} :: \frac{Q-P}{R-Q}$. From the product rule follows that the product of $R-P$ and $R-Q$ is known. So is their difference, which allows application of *Elements* II.6.

#15(391) Same proportion, $P$ unknown. Product rule and *Elements* II.5.

---*Modus alius proportionis*---

#16 $\frac{R}{P} :: \frac{(R-Q)+(Q-P)}{Q-P}$, $Q$ unknown. $\frac{R}{P} :: \frac{R-P}{Q-P}$, a linear problem.

#17 Same proportion, $R$ unknown.[27] $\frac{R-P}{P} :: \frac{(R-P)-(Q-P)}{Q-P}$, whence *permutatim* $\frac{R-P}{R-Q} :: \frac{P}{Q-P}$, a linear problem.

#18 Same proportion, $P$ unknown. *Eversim* (but Fibonacci writes "you permutate") $\frac{R}{R-P} :: \frac{R-P}{R-Q}$. The product rule gives $R-P$, whence $P$.

---*Incipit differentia tercia in proportione trium numerorum*---

[27] By error, the text has *minor numerus .a.d.*, but the calculation proceeds from the premise *major numerus .a.b.*, corresponding to $R$.

#19 No question but the observation that if $\frac{R}{Q} :: \frac{R-Q}{Q-P}$, then $P$, $Q$ and $R$ are in continued proportion—namely because $Q$ must be the same part of $R$ as $P$ of $Q$. This quasi-arithmetical approach (and the geometric line argument used to support it) is similar to what we encountered in the *Liber mahameleth*, p. $201_{10}$ (above, p. 18).

#20[(392)] $\frac{R}{Q} :: \frac{Q-P}{R-Q}$, $Q$ unknown. $\frac{Q}{R} :: \frac{R-Q}{Q-P}$, whence $\frac{Q+R}{R} :: \frac{R-P}{Q-P}$. The product rule and an addition allows the use of *Elements* II.6.

#21 Same proportion, $R$ unknown. The product rule and *Elements* II.6 give $R$.

#22 Same proportion, $P$ unknown. The product rule gives $Q-P$.

---*Modus proportionis in tribus numeris*---

#23 $\frac{R}{Q} :: \frac{(R-Q)+(Q-P)}{R-Q}$, $Q$ unknown. *Permutatim* and *conjunctim* $\frac{R+(R-P)}{R-P} :: \frac{R}{R-Q}$. From the product rule follows $R-Q$.

#24 Same proportion, $R$ unknown. The argument is corrupt, claiming that the proportion can be transformed into $\frac{Q}{R} :: \frac{R}{Q-P}$. The product rule and *Elements* II.5 would have led directly to a correct solution.

#25[(393)] Same proportion, $P$ unknown. $R-P$ follows from the product rule.

---*Modus alius proportionibus in tribus numeris*---

#26 $\frac{R}{Q} :: \frac{R-P}{Q-P}$, $Q$ unknown. *Disjunctim* $\frac{R-Q}{Q} :: \frac{R-Q}{Q-P}$. Since $P$ is a number (i.e., not 0, "zephirum, hoc est nihil"), $R$ must equal $Q$. Alternatively, the proportion is transformed *permutatim* into $\frac{R}{R-P} :: \frac{Q}{Q-P}$, and $R$ is posited to be 8, $P$ to be 2, from which is derived that $Q$ must equal $R$. Finally it is observed that even with this transformation the numerical position for $P$ is superfluous.

---*Modus alius proportionis in tribus numeris*---

#27 $\frac{Q}{P} :: \frac{R-Q}{Q-P}$, $Q$ unknown.[28] Instead of transforming *ex aequa* $\frac{Q}{P} :: \frac{Q+(R-Q)}{P+(Q-P)}$, i.e., $\frac{Q}{P} :: \frac{R}{Q}$, Fibonacci prefers to combine transformations *permutatim* ($\frac{R-Q}{Q} :: \frac{Q-P}{P}$) and *conjunctim* ($\frac{Q}{P} :: \frac{R}{Q}$, that is, producing the same outcome). From the product rule, $Q$ is found as $\sqrt{(PR)}$.

#28 Same proportion, $R$ unknown. Fibonacci uses the transformed

---

[28] The text says *ignotus primus numerus .a.g.*, but *ag* is actually the second, that is, $Q$.

proportion from #27 to find $R$ as $Q^2/P$.

#29 Same proportion. It is pointed out that if $Q$ is known (the example being $Q$ = 12), then either of the others can be chosen freely, the third number following (via $\frac{Q}{P} :: \frac{R}{Q}$) from division.

---*Modus alius proportionis in tribus numeris*---

#30(394) $\frac{Q}{P} :: \frac{R-P}{Q-P}$, $Q$ unknown. The product rule allows application of *Elements* II.6.

#31 Same proportion, $R$ unknown, $R-P = {}^{Q\cdot[Q-P]}/_{P}$ .

#32 Same proportion, $P$ unknown. *Eversim* $\frac{Q}{Q-P} : \frac{R-P}{R-Q}$. The product rule allows application of *Elements* II.6.

#33 It is then asserted that if one of the numbers is known in this proportion, the others can be found. What is actually shown (and obviously meant) is that if one is known, another one can be chosen *ad libitum*, and a third determined so as to fit.

---*Modus alius proportionis in tribus numeris*---

#34 $\frac{Q}{P} :: \frac{Q-P}{R-Q}$, $Q$ unknown. *Conjunctim* $\frac{Q+P}{P} :: \frac{R-P}{R-Q}$, which (via a trick necessitated by the line representation) allows application of *Elements* II.5.

#35 Same proportion, $R$ unknown. $R-Q = \frac{P\cdot(Q-P)}{Q}$.

#36(395) Same proportion, $P$ unknown. The product rule allows application of *Elements* II.5.

#37 $\frac{Q}{P} :: \frac{R-P}{R-Q}$. Since *eversim* $\frac{Q}{Q-P} :: \frac{R-P}{(R-P)-(R-Q)}$, i.e., $\frac{Q}{Q-P} :: \frac{R-P}{Q-P}$, this is only possible if $Q = R-P$ or, as Fibonacci prefers, $P = R-Q$. From this, any one of the numbers can be found if the other two are known.

---*Modus ultimus proportionis in tribus numeris*---

#38 Same proportion, $P+Q+R$ given. For three numbers $p$, $q$ and $r$ fulfilling the condition, multiply each of them by ${}^{P+Q+R}/_{p+q+r}$ (a scaling trick that is already used in Chapter 12, Section 2).

#39–50 consider four numbers in proportion, $\frac{P}{Q} :: \frac{R}{S}$. The underlying alphabetic order is still $a$, $b$, $g$, $d$, ….

---*Incipit de proportione quattuor numerorum*---

#39 From $\frac{P}{Q} :: \frac{R}{S}$ follows $\frac{Q}{P} :: \frac{S}{R}$ and $\frac{R}{P} :: \frac{S}{Q}$. From the product rule $PS = QR$, any one of the numbers can be found from the others.

#40 $P+Q$, $R$ and $S$ known. $\frac{P+Q}{Q}$ :: $\frac{R+S}{S}$, whence $Q$.
#41 $R+S$, $P$ and $Q$ are known. Similarly
#42 $P+R$, $Q$ and $S$ known. $\frac{Q+S}{S}$ :: $\frac{P+R}{R}$, whence $R$.
#43 $Q+S$, $P$ and $R$ are known. Similarly

-------

#44[396] $Q+R$, $P$ and $S$ known. The product rule allows application of *Elements* II.5.

#45 Similarly if $P+S$, $Q$ and $R$ are known. Illustrated by an example involving *rotuli* (a weight unit) and *bizantii* and their sum.

------

#46 $P–Q$, $R$ and $S$ known. $\frac{P-Q}{Q}$ :: $\frac{R-S}{R}$, whence $Q$.
#47 $R–S$, $P$ and $Q$ known. Similarly $\frac{P-Q}{P-Q}$ :: $\frac{R-S}{R}$, whence $R$.
#48 $P–R$, $Q$ and $S$ known. $\frac{P}{R}$ :: $\frac{Q}{S}$, $\frac{P-R}{R}$ :: $\frac{Q-S}{S}$, whence $R$.
#49 $P–S$, $Q$ and $R$ known. The product rule allows application of *Elements* II.6.

------

#50[397] $P^2+Q^2$, $R$ and $S$ known. Jumps directly (in a numerical example) to the proportion $\frac{P^2+Q^2}{Q^2}$ :: $\frac{R^2+S^2}{S^2}$. Once more, this echoes what we have encountered in the *Liber mahameleth*.

The Latin alphabetic order of #1–3 suggests that this group comes from Fibonacci's own pen, at least in its final redaction. The purely Greek or Arabic order of #7–50 suggest a less edited borrowing. The mixed usage of #4–5 (there is no lettered diagram for #6) is most likely to reflect that even these were borrowed, but the calculations reconstructed or made anew by Fibonacci.

The alphabetic order alone only tells us that the source was not Latin; however, the various points of contact of #6–50 with the *Liber mahameleth* suggest the Iberian area and hence al-Andalus.

We shall return to the global project inherent in #7–38 (there is indeed a striking theoretical project); but first we shall look at a few interesting passages from Sections 2 and 3. Section 2 is told to deal with "questions pertaining to geometry" (p. 397), which again is only partially true.

Those of the questions from Section 2 that concern us here are not geometrical. Instead they deal with commercial travels with composite gain. The first of them (p. 399) is in itself too simple to tell us much—but it is a background to those that follow: Somebody goes to one place of trade with 100 £ and earns, and afterwards earns proportionally in another place, and then has a total of 200 £. A continued proportion (represented by lettered line segments) shows the possession after the first travel to be $\sqrt{(100 \cdot 200)} \approx$ £ 141, s. 8, d. $5\frac{1}{8}$.

The next case (p. 399) is more tricky. The initial capital is still 100 £, but after the first travel a partner invests 100 £ in the enterprise, and after the second travel the total amounts to 299 £. This is expressed in the proportion (represented by lines) $\frac{100}{Q} :: \frac{Q+100}{299}$. From the product rule and *Elements* II.6 (still unidentified) follows the solution $Q = 130$ £. Interchange of left and right would reduce this to case #49 from Section 1, but Fibonacci does not make the connection. In any case, the use of proportion theory and of the unidentified Euclidean theorem instead of standard algebra[29] indicates a close link both to Section 1 and to the *Liber mahameleth*—a link which is not produced intentionally by Fibonacci, since he does not seem to notice it.

There are two more problems about repeated travels with gain; they also involve manipulation of proportions, but tell us nothing remarkable.

Section 3 is dedicated to "the solution of certain questions according to the method of algebra and almuchabala, that is, proportion and restoration" (p. 406). This presents us with an enigma. From Gerard's translation of al-Khwārizmī's *Algebra* Fibonacci knew that one of the techniques of the discipline was "restoration."[30] For the other, Gerard's text gave him no certain cues, and he himself only uses *oppositio* / *opponere* (the normal counterparts of *muqabalah* / *qabila*) thrice, and furthermore in the (probably original) sense of confronting the two sides of an equation.

[29] Admittedly, algebra has not been introduced at this point; but nothing asks for these problems to be dealt with in a geometry section.

[30] Fibonacci copied from this work and must hence have known it well, see [Miura 1981].

Fibonacci therefore had two slots for one operation—and guessed wrongly.[31] But from where did he get the idea that "proportion" was an essential ingredient? Proportions only play a minor role in what follows, so a description of his own approach does not seem to be behind the idea. As far as I know, no other obvious explanation is on offer—but if Fibonacci knew something similar to the blend of algebra and proportion theory in the *Liber mahameleth*, he would at least have a reason for his bad guess.

Among the appearances of proportions within the section, at least one seems to belong to the types we know from Section 1 and from the *Liber mahameleth* (others are possible but not sufficiently characteristic). On p. 423, Fibonacci asks for a number which, when $^1/_3$ of it and 6 more are removed and the remainder multiplied by itself, yields twice the original number—in symbols,

$$(x-{}^1/_3x-6)^2 = 2x \ .$$

We are told that this could be found by algebra, but that is not done. Instead, Fibonacci makes use of a line diagram and transforms the data into a proportion which in symbols becomes

$$\frac{\frac{2}{3}x}{x-\frac{1}{3}x-6} = \frac{x-\frac{1}{3}x-6}{3} \ .$$

*Disjunctim*, this allows him to apply *Elements* II.6 (unidentified once again). The underlying alphabetic order is *a*, *b*, *g*, *d*, which is unusual in this section.

So, Sections 2 and 3 contain scattered traces of that same influence which dominates Section 1, and which has remarkable similarity to what we know from the *Liber Mahameleth*. But as promised we shall have another look at Section 1, whose #7–38 turn out to present us with an unparalleled theoretical investigation.

Pappos's *Collection* III, "Concerning plane and solid geometrical

[31] His Arabic, learned presumably during a relatively short period in his boyhood ("some days," he claims in the introduction to the *Liber abbaci*, p. 1) and afterwards perhaps only improved in the course of commercial interactions, must have been less perfect than often assumed.

problems” [ed., trans. Hultsch 1876: I, 30–177], largely deals with *means*. Propositions xii–xxiii (pp. 70–105), in particular, deal with a set of 10 means between two magnitudes—the three original ones (arithmetical, geometrical, harmonic), which are “very useful for dealing with ancient writings”; with three more that were added by early authors, and finally with four joined to them by “the more recent” (pp. 84–85). The last part of Nicomachos’ *Introduction to Arithmetic* (II.xii–xxviii [ed. Hoche 1866: 122–144], trans. [d’Ooge 1926: 266–284]) deals with almost the same, though now between numbers. The two are discussed in parallel by Thomas Heath [1921: 86–89], who observes that one of Nicomachos’s means is omitted by Pappos, and vice versa.

As it turns out, the sequence #7–38 is very closely related to this ancient classification and theory of means: for two numbers $P$ and $R$, and the mean $Q$, it is shown how knowledge of any two of them allows us to find the third. This table illustrates it:

| | Pappos | Nicomachos | *Liber abbaci* |
|---|---|---|---|
| $\frac{R-Q}{Q-P} :: \frac{R}{R}$ (arithmet.) | P1 | N1 | |
| $\frac{R-Q}{Q-P} :: \frac{R}{Q}$ or $\frac{R-Q}{Q-P} :: \frac{Q}{P}$ | P2 | N2 | #27–29 |
| $\frac{R-Q}{Q-P} :: \frac{R}{P}$ | P3 | N3 | #7–9 |
| $\frac{R-Q}{Q-P} :: \frac{P}{R}$ | P4 | N4 (but inverted) | #10–12 (inverted) |
| $\frac{R-Q}{Q-P} :: \frac{P}{Q}$ | P5 | N5 (but inverted) | #34–36 (inverted) |
| $\frac{R-Q}{Q-P} :: \frac{Q}{R}$ | P6 | N6 (but inverted) | #20–22 (inverted) |
| $\frac{R-P}{Q-P} :: \frac{R}{P}$ | absent | N7 | #16–18 |
| $\frac{R-P}{R-Q} :: \frac{R}{P}$ | P9 | N8 | #13–15 |
| $\frac{R-P}{Q-P} :: \frac{Q}{P}$ | P10 | N9 | #30–32 |

| | | | |
|---|---|---|---|
| $\frac{R-P}{R-Q} :: \frac{Q}{P}$ | P7 | N10 | #37–38 |
| $\frac{R-P}{R-Q} :: \frac{R}{Q}$ | P8 | absent | #23–25 |

I have followed Nicomachos's order, as being the most relevant. As we see, indeed, Fibonacci agrees with Nicomachos and Boethius and not with Pappos in the cases 4–6, having $\frac{R}{P} :: \frac{Q-P}{R-Q}$ instead of $\frac{R-Q}{Q-P} :: \frac{P}{R}$, etc. We may thus assume that the ultimate inspiration is Nicomachos, not Pappus. This is not astonishing, Nicomachos was well known in the Arabic as well as in the Latin world (in the latter in Boethius's translation[32]). My initial quotation from Djebbar also pointed to a tradition starting from Thābit ibn Qurra's translation of Nicomachos.

There are obvious differences, however. Firstly, the *Liber abbaci* contains no counterpart of the arithmetical mean, whose expression as a proportion is indeed next to ridiculous—the only sensible definition is $R–Q = Q–P$; whoever made Nicomachos' material the object of systematic theoretical exploration has seen that. On the other hand, the case omitted by Nicomachos but discussed by Pappos is included. This is not evidence of contamination, a competent mathematician going through all the cases would observe that it had its place. Similarly he observes that Fibonacci's #26 ought to be there, at least once we have given up the idea that $Q$ is a genuine mean and therefore $P<Q<R$; and as we see, Fibonacci does not speak of means (his reference to the "middle" number is only occasional, and serves as identification only), for which reason we may assume that his source did not either. Finally, the treatment of means is extended by a similar investigation of four numbers in proportion, which because of the alphabetic ordering is likely also to represent a borrowing, plausibly from the same source.

All in all: the vicinity of the methods of Fibonacci's #7–50 to those of the *Liber mahameleth*, together with the uniqueness of the two texts, makes

---

32 Boethius presentation (*De institutione arithmetica*, ed. [Friedlein 1867: 140–169]) agrees exactly with Nicomachos in all relevant respects.

it reasonably certain that even Fibonacci's source came, if not from the same hand then at least from the same environment as the theoretical exploration of the possibilities offered by *muʿāmalāt* mathematics.

On the other hand, the absence of anything in known Ibero-Latin texts similar to Fibonacci's Chapter 15 Section 1 makes it implausible that any of the two came from the ambience of translators into Latin and the users of their translations—even more implausible, we may say, than the idea is in itself that the *Liber mahamalech* should have been produced by Gundisalvi or an associate of his.

### Jacopo's quasi-algebra

A vernacular source of interest is Jacopo da Firenze's *Tractatus algorismi*—more precisely a sequence of problems dealing with the wages of the manager of a *fondaco* over three or four years, supposed to increase geometrically.[33] Designating the four salaries *a*, *b*, *d* and *e*, we may express the problems thus:

(1) $a+d = 20$, and $b = 8$.

(2) $a = 15$, $e = 60$.

(3) $a+e = 90$, $b+d = 60$.

(4) $a+d = 20$, $b+e = 30$

All solutions are given as numerical prescriptions only, there is neither *cosa-censo* algebra nor line diagrams (the latter are never used by Jacopo). But is it easy to change the descriptions unambiguously into formulae.

Problem (1) is solved by in this way:[34]

$$a = \frac{a+d}{2} - \sqrt{(\frac{a+d}{2})^2 - ad} \quad \text{and} \quad d = \frac{a+d}{2} + \sqrt{(\frac{a+d}{2})^2 - ad} .$$

Obviously, the product rule gives $ad = b^2$, after which the rule

---

[33] Once more only in MS Vat. Lat. 4826 [ed. trans. Høyrup 2007: 326–331], separated from the algebra section proper by only an alligation problem dealing with grain. Cf. above, note ?.

[34] I discover a misprint in my analysis in [Høyrup 2007: 116]—in the formula to the right, *b* should be *d* (as here).

corresponding to *Elements* II.5 gives these.

Problem (2) is solved as

$$b = \sqrt[3]{\frac{d}{a} \cdot a^3} \quad , \quad d = \sqrt[3]{(\frac{d}{a})^2 \cdot a^3} \quad ,$$

which asks for nothing but familiarity with the theory of continued proportions (duly generalized so as to accept irrational radicals). If we think in terms of a factor of proportionality *s*, as we know it from the *Liber mahameleth*, it can be solved as the algebraic problem "cubes equal number," for which Jacopo has presented the rule earlier on [ed. Høyrup 2007: 320]—but no such link is made, and it is doubtful whether Jacopo really understood algebra beyond the second degree (he offers no examples, only rules).

Problem (3) is more astonishing. It is solved by means of the formula

$$a{\cdot}e = b{\cdot}d = \frac{(b+d)^3}{3(b+d)+(a+e)} \; ,$$

which is easily justified if we think in terms of a factor of proportionality *s*, since then

$$\frac{(b+d)^3}{3(b+d)+(a+e)} = \frac{a^3s^3(1+s)^3}{a\,(3s+3s^2+1+s^3)} = a^2s^3 = a \cdot as^3 = as \cdot as^2 \; .$$

How it had been found is another matter. In any case, once $a{\cdot}e = b{\cdot}d$ are found, $a+e$ and $b+d$ being already given, all four can be found (and are indeed found) by means of the rule corresponding to *Elements* II.5

Without identifying it, problem (4) finds the factor of proportionality *s* as $(b+e)/(a+d)$, whence

$$a = \frac{a+d}{1+p^2} \; , \;\; d = (a+d)–a \; , \;\; b = \frac{b+e}{1+p^2} \; , \;\; e = (b+e)–b \; .$$

Since the factor of proportionality is actually found and used here, we may be confident that it also underlies the solutions of problems (2) and (3). Already this, together with the use of a rule derived from *Elements* II.5, already suggests kinship with *Liber mahameleth* and *Liber abbaci*, chapter

15. The idea to use a (pseudo-)practical problem as a starting point or pretext for deeper study, on the other hand, recalls what was done to the unknown heritage and the treatment of commercial problems in the *Liber mahameleth*. In the absence of anything similar in earlier extant sources[35] we may therefore assume with fair certainty that even this group of problems has been drawn from the same al-Andalus reservoir as the three Latin sources that were discussed above.

Nothing is certain in this world. Even the theory that Shakespeare's works were not written by Shakespeare but by somebody else also called Shakespeare has its proponents, as we know. But with that proviso I think we may conclude that twelfth-century *al-Andalus* was the home to astronomer-mathematicians who systematically developed *theory* or something close to that from the questions of simpler, non-theoretical mathematics—be it the puzzling unknown heritage, be it commercial arithmetic, be it Nicomachos's list of means, be it finally the riddle about geometricaly increasing wages.

## References


Allard, André (ed., trans.), 1981. Maxime Planude, *Le Grand calcul selon les Indiens*. Louvain-la-Neuve: [Faculté de Philosophie et Lettres].

Arrighi, Gino (ed.), 1964. Paolo Dell'Abaco, *Trattato d'aritmetica*. Pisa: Domus Galilæana.

Baur, Ludwig (ed.), 1903. Dominicus Gundissalinus, *De divisione philosophiae*. (Beiträge zur Geschichte der Philosophie des Mittelalters. Texte und Untersuchngen, 4:2–3). Münster: Aschendorffsche Buchhandlung.

Boncompagni, Baldassare (ed.), 1857. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. I. Il *Liber abbaci* di Leonardo Pisano. Roma: Tipografia delle Scienze Matematiche e Fisiche.

Burnett, Charles, 2002. "John of Seville and John of Spain, a *mise au point*". *Bulletin*


[35] Scattered later sources until Pacioli, Tartaglia and Cardano do contain some similar problems, with variations that suggest Jacopo not to have been the only channell through whom the problem type reached the Italian abbacus environment—see [Høyrup 2007: 117–120] and [Kichenassamy 2015].

*de philosophie medievale* **44**, 59–78.
Burnett, Charles, Ji-Wei Zhao & Kurt Lampe, 2007. "The Toledan *Regule:* a Twelfth-Century Arithmetical Miscellany". *SCIAMUS* **8**, 141–231.
Busard, H. L. L., 2005. *Campanus of Novara and Euclid's Elements*. 2 vols. Stuttgart: Franz Steiner.
Cardano, Girolamo, 1539. *Practica arithmetice, et mensurandi singularis*. Milano: Bernardini Calusco.
Caunedo del Potro, Betsabé, & Ricardo Córdoba de la Llave (eds), 2000. *El arte del alguarismo*. Un libro castellano de aritmética comercial y de ensayo de moneda del siglo XIV. (Ms. 46 de la Real Colegiato de San Isidoro de León). Salamanca: Junta de Castilla y León, Consejeria de Educación y Cultura.
d'Ooge, Martin Luther (trans.), 1926. Nicomachus of Gerasa, *Introduction to Arithmetic*. With Studies of Greek Arithmetic by Frank Egleston Robbins and Louis Charles Karpinski. New York & London: MacMillan.
Djebbar, Ahmed, 1993. "Deux mathématiciens peu connus de l'Espagne du XI[e] siècle: al-Maʾtaman and Ibn Sayyid", pp. 79–91 *in* M. Folkerts & J. P. Hogendijk (eds), *Vestigia Mathematica. Studies in Medieval and* Early Modern Mathematics in Honour of H. L. L. Busard. Amsterdam & Atlanta: Rodopi.
Euler, Leonhard, 1774. *Élémens d'algebre*. I. *De l'analyse déterminée*. II. *De l'analyse indéterminée*. Lyon: Jean-Marie Bruyset.
Folkerts, Menso, 2006. *The Development of Mathematics in Medieval Europe: The Arabs, Euclid, Regiomontanus*. Aldershot: Ashgate.
Friedlein, Gottfried (ed.), 1867. Anicii Manlii Torquati Severini Boetii *De institutione arithmetica* libri duo. *De institutiune musica* libri quinque. Accedit *Geometria* quae fertur Boetii. Leipzig: Teubner.
Gonzales Palencia, Angel (ed., trans.), 1953. Al-Fārābī, *Catálogo de las ciencias*. Edición y traduccion Castellana. Segunda edición. Madrid: Consejo Superior de Investigaciones Científicas, Instituto Miguel Asín.
Heath, Thomas L., 1921. *A History of Greek Mathematics*. 2 vols. Oxford: Clarendon Press.
Heiberg, J. L. (ed., trans.), 1912. Heronis *Definitiones* cum variis collectionibus. Heronis quae feruntur *Geometrica*. (Heronis Alexandrini Opera quae supersunt omnia, IV). Leipzig: Teubner.
Hoche, Richard (ed.), 1866. Nicomachi Geraseni Pythagorei *Introductionis arithmeticae* libri II. Leipzig: Teubner.
Høyrup, Jens, 1987. "The Formation of 'Islamic Mathematics'. Sources and Conditions". *Science in Context* **1**, 281–329.
Høyrup, Jens, 2007. *Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture*. Basel etc.: Birkhäuser.
Høyrup, Jens, 2008. "The 'Unknown Heritage': Trace of a Forgotten Locus of Mathematical Sophistication". *Archive for History of Exact Sciences* **62**, 613–654.
Høyrup, Jens, 2011. "«Proportions» in the *Liber abbaci*", pp. 83–100 *in* Sabine

Rommevaux, Philippe Vendrix & Vasco Zara (eds), *Proportions: Science – Musique – Peinture & Architecture*. Actes du LI[e] Colloque International d'Études Humanistes, 30 juin – 4 juillet 2008. Turnhout: Brepols, 2011.

Høyrup, Jens, 2015. [Review of Anne-Marie Vlasschaert (ed.), *Liber Mahameleth*. (Boethius). Wiesbaden: Franz Steiner, 2010]. *Speculum* **90**, 1177–1178.

Høyrup, Jens, 2016. "Which Kind of Mathematics Was Known and Referred to by Those Who Wanted to Integrate Mathematics in «Wisdom» – Neopythagoreans and Others?" *AIMS Mathematics* **1**, 24–42.

Hultsch, Friedrich (ed., trans.), 1876. Pappi Alexandrini *Collectionis* quae supersunt. E libris manu scriptis edidit et commentariis instruxit Fridericus Hultsch. I, II, III.1-2. Berlin: Weidmann, 1876–78.

Kichenassamy, Satyanad, 2015. "Continued Proportions and Tartaglia's Solution of Cubic Equations". *Historia Mathematica* **42**, 407–435.

Lorch, Richard P., 1973. "Jābir ibn Aflaḥ", pp. 37–39 *in Dictionary of Scientific Biography*, vol. VII. New York: Scribner.

Miura, Nobuo, 1981. "The Algebra in the *Liber abaci* of Leonardo Pisano". *Historia Scientiarum* **21**, 57–65.

Rashed, Roshdi (ed., trans.), 2013. Abū Kāmil, *Algèbre et analyse Diophantienne*. Berlin & Boston: de Gruyter.

Rebstock, Ulrich, 1993. *Die Reichtümer der Rechner (Ġunyat al-Ḥussāb) von Aḥmad b. Ṯabāt (gest. 631/1234). Die Araber – Vorläufer der Rechenkunst*. Walldorf-Hessen: Verlag für Orientkunde Dr. H. Vorndran.

Rebstock, Ulrich (ed., trans.), 2001. ᶜAlī ibn al-Ḫiḍr al-Qurašī (st. 459 / 1067), *At-taḏkira bi-uṣūl al-ḥisāb wa l'farāʾiḍ* (*Buch über die Grundlagen der Arithmetik und der Erbteilung*). Frankfurt a.M.: Institute for the History of Arabic-Islamic Science at the Johann Wolfgang Goethe University.

Sabra, A. I., 1987. "The Appropriation and Subsequent Naturalization of Greek Science in Medieval Islam: A Preliminary Statement". *History of Science* **25**, 223–243.

Sesiano, Jacques, 1988. "Le Liber Mahamaleth, un traité mathématique latin composé au XII[e] siècle en Espagne", pp. 69–98 *in Histoire des Mathématiques Arabes*. Premier colloque international sur l'histoire des mathématiques arabes, Alger, 1.2.3 décembre 1986. Actes. Alger: La Maison des Livres.

Sesiano, Jacques, 2014. *The* Liber mahamaleth: *A Twelfth-Century Mathematical Treatise*. Heidelberg etc.: Springer.

Spiesser, Maryvonne (ed.), 2003. *Une arithmétique commerciale du XV[e] siècle*. Le *Compendy de la praticque des nombres* de Barthélemy de Romans. Turnhout: Brepols.

Taisbak, Christian Marinus (ed., trans.), 2003. Δεδομενα*: Euclid's* Data, *or, The Importance of Being Given*. Copenhagen: Museum Tusculanum.

Vlasschaert, Anne-Marie (ed.), 2010. *Liber Mahamaleth*. Wiesbaden: Franz Steiner.